

# NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR CONVEX FUNCTIONS WITH APPLICATIONS

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**ABSTRACT.** In this paper, some new inequalities of the Hermite-Hadamard type for functions whose modulus of the derivatives are convex and applications for special means are given. Finally, some error estimates for the trapezoidal formula are obtained.

## 1. INTRODUCTION

A function  $f : I \rightarrow \mathbb{R}$  is said to be convex function on  $I$  if the inequality

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y),$$

holds for all  $x, y \in I$  and  $\alpha \in [0, 1]$ .

One of the most famous inequality for convex functions is so called Hermite-Hadamard's inequality as follows: Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a convex function defined on the interval  $I$  of real numbers and  $a, b \in I$ , with  $a < b$ . Then :

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

In [3], the following theorem which was obtained by Dragomir and Agarwal contains the Hermite-Hadamard type integral inequality.

**Theorem 1.** *Let  $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$ ,  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|$  is convex on  $[a, b]$ , then the following inequality holds:*

$$(1.2) \quad \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}.$$

In [4] Kırmacı, Bakula, Özdemir and Pečarić proved the following theorem.

**Theorem 2.** *Let  $f : I \rightarrow \mathbb{R}$ ,  $I \subset \mathbb{R}$ , be a differentiable function on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$ ,  $a < b$ . If  $|f'|^q$  is concave on  $[a, b]$  for some  $q > 1$ , then:*

$$(1.3) \quad \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \left( \frac{b-a}{4} \right) \left[ \frac{q-1}{2q-1} \right]^{\frac{q-1}{q}} \left( \left| f' \left( \frac{a+3b}{4} \right) \right| + \left| f' \left( \frac{3a+b}{4} \right) \right| \right).$$

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For recent results and generalizations concerning Hermite-Hadamard's inequality see [1]-[4] and the references therein.

## 2. THE NEW HERMITE-HADAMARD TYPE INEQUALITIES

In order to prove our main theorems, we first prove the following lemma:

**Lemma 1.** *Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  (the interior of  $I$ ), where  $a, b \in I$  with  $a < b$ . If  $f' \in L[a, b]$ , then the following inequality holds:*

$$\begin{aligned} & \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \\ &= \frac{(x-a)^2}{b-a} \int_0^1 (t-1) f'(tx + (1-t)a) dt + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) f'(tx + (1-t)b) dt. \end{aligned}$$

*Proof.* We note that

$$\begin{aligned} I &= \frac{(x-a)^2}{b-a} \int_0^1 (t-1) f'(tx + (1-t)a) dt \\ &\quad + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) f'(tx + (1-t)b) dt. \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} I &= \frac{(x-a)^2}{b-a} \left[ (t-1) \frac{f(tx + (1-t)a)}{x-a} \Big|_0^1 - \int_0^1 \frac{f(tx + (1-t)a)}{x-a} dt \right] \\ &\quad + \frac{(b-x)^2}{b-a} \left[ (1-t) \frac{f(tx + (1-t)b)}{x-b} \Big|_0^1 + \int_0^1 \frac{f(tx + (1-t)b)}{x-b} dt \right] \\ &= \frac{(x-a)^2}{b-a} \left[ \frac{f(a)}{x-a} - \frac{1}{(x-a)^2} \int_a^x f(u) du \right] \\ &\quad + \frac{(b-x)^2}{b-a} \left[ -\frac{f(b)}{x-b} + \frac{1}{(x-b)^2} \int_b^x f(u) du \right] \\ &= \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du. \end{aligned}$$

□

Using the Lemma 1 the following result can be obtained.

**Theorem 3.** *Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|$  is convex on  $[a, b]$ , then the following inequality holds:*

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \frac{(x-a)^2}{b-a} \left[ \frac{|f'(x)| + 2|f'(a)|}{6} \right] + \frac{(b-x)^2}{b-a} \left[ \frac{|f'(x)| + 2|f'(b)|}{6} \right] \end{aligned}$$

for each  $x \in [a, b]$ .

*Proof.* Using Lemma 1 and taking the modulus, we have

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)a)| dt \\ & \quad + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)b)| dt. \end{aligned}$$

Since  $|f'|$  is convex, then we get

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) [t |f'(x)| + (1-t) |f'(a)|] dt \\ & \quad + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) [t |f'(x)| + (1-t) |f'(b)|] dt \\ & = \frac{(x-a)^2}{b-a} \left[ \frac{|f'(x)| + 2|f'(a)|}{6} \right] + \frac{(b-x)^2}{b-a} \left[ \frac{|f'(x)| + 2|f'(b)|}{6} \right] \end{aligned}$$

which completes the proof.  $\square$

**Corollary 1.** In Theorem 3, if we choose  $x = \frac{a+b}{2}$ , we obtain

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{b-a}{12} \left( |f'(a)| + \left| f' \left( \frac{a+b}{2} \right) \right| + |f'(b)| \right).$$

**Remark 1.** In Corollary 1, using the convexity of  $|f'|$ , we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|)$$

which is the inequality in (1.2).

**Theorem 4.** Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|^{\frac{p}{p-1}}$  is convex on  $[a, b]$  and for some fixed  $p > 1$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}} \\ & \quad \times \left[ \frac{(x-a)^2 [|f'(a)|^q + |f'(x)|^q]^{\frac{1}{q}} + (b-x)^2 [|f'(x)|^q + |f'(b)|^q]^{\frac{1}{q}}}{b-a} \right] \end{aligned}$$

for each  $x \in [a, b]$  and  $q = \frac{p}{p-1}$ .

*Proof.* From Lemma 1 and using the well-known Hölder integral inequality, we have

$$\begin{aligned}
& \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\
& \leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)a)| dt \\
& \quad + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)b)| dt \\
& \leq \frac{(x-a)^2}{b-a} \left( \int_0^1 (1-t)^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\
& \quad + \frac{(b-x)^2}{b-a} \left( \int_0^1 (1-t)^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Since  $|f'|^{\frac{p}{p-1}}$  is convex, by the Hermite-Hadamard's inequality, we have

$$\int_0^1 |f'(tx + (1-t)a)|^q dt \leq \frac{|f'(a)|^q + |f'(x)|^q}{2}$$

and

$$\int_0^1 |f'(tx + (1-t)b)|^q dt \leq \frac{|f'(b)|^q + |f'(x)|^q}{2},$$

so

$$\begin{aligned}
& \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\
& \leq \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}} \\
& \quad \times \left[ \frac{(x-a)^2 [|f'(a)|^q + |f'(x)|^q]^{\frac{1}{q}} + (b-x)^2 [|f'(x)|^q + |f'(b)|^q]^{\frac{1}{q}}}{b-a} \right]
\end{aligned}$$

which completes the proof.  $\square$

**Corollary 2.** In Theorem 4, if we choose  $x = \frac{a+b}{2}$ , we obtain

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) du \right| \\
& \leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}} \\
& \quad \times \left[ \left( |f'(a)|^q + \left| f' \left( \frac{a+b}{2} \right) \right|^q \right)^{\frac{1}{q}} + \left( |f'(b)|^q + \left| f' \left( \frac{a+b}{2} \right) \right|^q \right)^{\frac{1}{q}} \right] \\
& \leq \frac{b-a}{2} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}} (|f'(a)| + |f'(b)|).
\end{aligned}$$

The second inequality is obtained using the following fact:  $\sum_{k=1}^n (a_k + b_k)^s \leq \sum_{k=1}^n (a_k)^s + \sum_{k=1}^n (b_k)^s$  for  $(0 \leq s < 1)$ ,  $a_1, a_2, a_3, \dots, a_n \geq 0$ ;  $b_1, b_2, b_3, \dots, b_n \geq 0$  with  $0 \leq \frac{p-1}{p} < 1$ , for  $p > 1$ .

**Theorem 5.** Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|^q$  is concave on  $[a, b]$ , for some fixed  $q > 1$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \left[ \frac{q-1}{2q-1} \right]^{\frac{q-1}{q}} \left[ \frac{(x-a)^2 |f'(\frac{a+x}{2})| + (b-x)^2 |f'(\frac{b+x}{2})|}{b-a} \right] \end{aligned}$$

for each  $x \in [a, b]$ .

*Proof.* As in Theorem 4, using Lemma 1 and the well-known Hölder integral inequality for  $q > 1$  and  $p = \frac{q}{q-1}$ , we have

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)a)| dt \\ & \quad + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)b)| dt \\ & \leq \frac{(x-a)^2}{b-a} \left( \int_0^1 (1-t)^{\frac{q}{q-1}} dt \right)^{\frac{q-1}{q}} \left( \int_0^1 |f'(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\ & \quad + \frac{(b-x)^2}{b-a} \left( \int_0^1 (1-t)^{\frac{q}{q-1}} dt \right)^{\frac{q-1}{q}} \left( \int_0^1 |f'(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since  $|f'|^q$  is concave on  $[a, b]$ , we can use the Jensen's integral inequality to obtain:

$$\begin{aligned} \int_0^1 |f'(tx + (1-t)a)|^q dt &= \int_0^1 t^0 |f'(tx + (1-t)a)|^q dt \\ &\leq \left( \int_0^1 t^0 dt \right) \left| f' \left( \frac{1}{\int_0^1 t^0 dt} \int_0^1 (tx + (1-t)a) dt \right) \right|^q \\ &= \left| f' \left( \frac{a+x}{2} \right) \right|^q. \end{aligned}$$

Analogously,

$$\int_0^1 |f'(tx + (1-t)b)|^q dt \leq \left| f' \left( \frac{b+x}{2} \right) \right|^q.$$

Combining all the obtained inequalities, we get

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \left[ \frac{q-1}{2q-1} \right]^{\frac{q-1}{q}} \left[ \frac{(x-a)^2 |f'(\frac{a+x}{2})| + (b-x)^2 |f'(\frac{b+x}{2})|}{b-a} \right] \end{aligned}$$

which completes the proof.  $\square$

**Remark 2.** In Theorem 5, if we choose  $x = \frac{a+b}{2}$ , we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \left[ \frac{q-1}{2q-1} \right]^{\frac{q-1}{q}} \left( \frac{b-a}{4} \right) \left( \left| f' \left( \frac{3a+b}{4} \right) \right| + \left| f' \left( \frac{a+3b}{4} \right) \right| \right) \end{aligned}$$

which is the inequality in (1.3).

**Theorem 6.** Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|^q$  is convex on  $[a, b]$ , for some fixed  $q \geq 1$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \frac{1}{2} \left( \frac{1}{3} \right)^{\frac{1}{q}} \left[ \frac{(x-a)^2 [|f'(x)|^q + 2|f'(a)|^q]^{\frac{1}{q}} + (b-x)^2 [|f'(x)|^q + 2|f'(b)|^q]^{\frac{1}{q}}}{b-a} \right] \end{aligned}$$

for each  $x \in [a, b]$ .

*Proof.* Suppose that  $q \geq 1$ . From Lemma 1 and using the well-known power-mean inequality, we have

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)a)| dt \\ & \quad + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)b)| dt \\ & \leq \frac{(x-a)^2}{b-a} \left( \int_0^1 (1-t) dt \right)^{1-\frac{1}{q}} \left( \int_0^1 (1-t) |f'(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\ & \quad + \frac{(b-x)^2}{b-a} \left( \int_0^1 (1-t) dt \right)^{1-\frac{1}{q}} \left( \int_0^1 (1-t) |f'(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since  $|f'|^q$  is convex, therefore we have

$$\begin{aligned} & \int_0^1 (1-t) |f'(tx + (1-t)a)|^q dt \\ & \leq \int_0^1 (1-t) [t|f'(x)|^q + (1-t)|f'(a)|^q] dt \\ & = \frac{|f'(x)|^q + 2|f'(a)|^q}{6} \end{aligned}$$

Analogously,

$$\int_0^1 (1-t) |f'(tx + (1-t)b)|^q dt \leq \frac{|f'(x)|^q + 2|f'(b)|^q}{6}.$$

Combining all the above inequalities gives the desired result.  $\square$

**Corollary 3.** *In Theorem 6, choosing  $x = \frac{a+b}{2}$  and then using the convexity of  $|f'|^q$ , we have*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \left( \frac{b-a}{8} \right) \left( \frac{1}{3} \right)^{\frac{1}{q}} \left[ \left( 2|f'(a)|^q + \left| f' \left( \frac{a+b}{2} \right) \right|^q \right)^{\frac{1}{q}} + \left( 2|f'(b)|^q + \left| f' \left( \frac{a+b}{2} \right) \right|^q \right)^{\frac{1}{q}} \right] \\ & \leq \left( \frac{3^{1-\frac{1}{q}}}{8} \right) (b-a) (|f'(a)| + |f'(b)|). \end{aligned}$$

**Theorem 7.** *Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|^q$  is concave on  $[a, b]$ , for some fixed  $q \geq 1$ , then the following inequality holds:*

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \frac{1}{2} \left[ \frac{(x-a)^2 |f'(\frac{x+2a}{3})| + (b-x)^2 |f'(\frac{x+2b}{3})|}{b-a} \right]. \end{aligned}$$

*Proof.* First, we note that by the concavity of  $|f'|^q$  and the power-mean inequality, we have

$$|f'(tx + (1-t)a)|^q \geq t|f'(x)|^q + (1-t)|f'(a)|^q.$$

Hence,

$$|f'(tx + (1-t)a)| \geq t|f'(x)| + (1-t)|f'(a)|,$$

so  $|f'|$  is also concave.

Accordingly, using Lemma 1 and the Jensen integral inequality, we have

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)a)| dt \\ & \quad + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)b)| dt \\ & \leq \frac{(x-a)^2}{b-a} \left( \int_0^1 (1-t) dt \right) \left| f' \left( \frac{\int_0^1 (1-t)(tx + (1-t)a) dt}{\int_0^1 (1-t) dt} \right) \right| \\ & \quad + \frac{(b-x)^2}{b-a} \left( \int_0^1 (1-t) dt \right) \left| f' \left( \frac{\int_0^1 (1-t)(tx + (1-t)b) dt}{\int_0^1 (1-t) dt} \right) \right| \\ & \leq \frac{1}{2} \left[ \frac{(x-a)^2 |f'(\frac{x+2a}{3})| + (b-x)^2 |f'(\frac{x+2b}{3})|}{b-a} \right]. \end{aligned}$$

□

**Corollary 4.** *In Theorem 7, if we choose  $x = \frac{a+b}{2}$ , we have*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \frac{b-a}{8} \left[ \left| f' \left( \frac{5a+b}{6} \right) \right| + \left| f' \left( \frac{a+5b}{6} \right) \right| \right]. \end{aligned}$$

### 3. APPLICATIONS TO SPECIAL MEANS

Recall the following means which could be considered extensions of arithmetic, logarithmic and generalized logarithmic from positive to real numbers.

(1) The arithmetic mean:

$$A = A(a, b) = \frac{a+b}{2}; \quad a, b \in \mathbb{R}$$

(2) The logarithmic mean:

$$L(a, b) = \frac{b-a}{\ln|b| - \ln|a|}; \quad |a| \neq |b|, \quad ab \neq 0, \quad a, b \in \mathbb{R}$$

(3) The generalized logarithmic mean:

$$L_n(a, b) = \left[ \frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)} \right]^{\frac{1}{n}}; \quad n \in \mathbb{Z} \setminus \{-1, 0\}, \quad a, b \in \mathbb{R}, \quad a \neq b$$

Now using the results of Section 2, we give some applications to special means of real numbers.

**Proposition 1.** *Let  $a, b \in \mathbb{R}$ ,  $a < b$ ,  $0 \notin [a, b]$  and  $n \in \mathbb{Z}$ ,  $|n| \geq 2$ . Then, for all  $p > 1$*

(a)

$$(3.1) \quad |A(a^n, b^n) - L_n^n(a, b)| \leq |n|(b-a) \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}} A(|a|^{n-1}, |b|^{n-1})$$

and

(b)

$$(3.2) \quad |A(a^n, b^n) - L_n^n(a, b)| \leq |n|(b-a) \frac{3^{1-\frac{1}{q}}}{4} A(|a|^{n-1}, |b|^{n-1}).$$

*Proof.* The assertion follows from Corollary 2 and 3 for  $f(x) = x^n$ ,  $x \in \mathbb{R}$ ,  $n \in \mathbb{Z}$ ,  $|n| \geq 2$ .  $\square$

**Proposition 2.** *Let  $a, b \in \mathbb{R}$ ,  $a < b$ ,  $0 \notin [a, b]$ . Then, for all  $q \geq 1$ ,*

(a)

$$(3.3) \quad |A(a^{-1}, b^{-1}) - L^{-1}(a, b)| \leq (b-a) \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}} A(|a|^{-2}, |b|^{-2})$$

and

(b)

$$(3.4) \quad |A(a^{-1}, b^{-1}) - L^{-1}(a, b)| \leq (b-a) \left( \frac{3^{1-\frac{1}{q}}}{4} \right) A(|a|^{-2}, |b|^{-2}).$$



*Proof.* The assertion follows from Corollary 2 and 3 for  $f(x) = \frac{1}{x}$ .  $\square$

#### 4. THE TRAPEZOIDAL FORMULA

Let  $d$  be a division  $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$  of the interval  $[a, b]$  and consider the quadrature formula

$$(4.1) \quad \int_a^b f(x) dx = T(f, d) + E(f, d)$$

where

$$T(f, d) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i)$$

for the trapezoidal version and  $E(f, d)$  denotes the associated approximation error.

**Proposition 3.** *Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$  and  $|f'|^{\frac{p}{p-1}}$  is convex on  $[a, b]$ , where  $p > 1$ . Then in (4.1), for every division  $d$  of  $[a, b]$ , the trapezoidal error estimate satisfies*

$$|E(f, d)| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{q}} \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^2}{2} (|f'(x_i)| + |f'(x_{i+1})|).$$

*Proof.* On applying Corollary 2 on the subinterval  $[x_i, x_{i+1}]$  ( $i = 0, 1, 2, \dots, n-1$ ) of the division, we have

$$\begin{aligned} & \left| \frac{f(x_i) + f(x_{i+1})}{2} - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x) dx \right| \\ & \leq \frac{(x_{i+1} - x_i)}{2} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{q}} (|f'(x_i)| + |f'(x_{i+1})|). \end{aligned}$$

Hence in (4.1) we have

$$\begin{aligned} \left| \int_a^b f(x) dx - T(f, d) \right| &= \left| \sum_{i=0}^{n-1} \left\{ \int_{x_i}^{x_{i+1}} f(x) dx - \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i) \right\} \right| \\ &\leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x) dx - \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i) \right| \\ &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{q}} \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^2}{2} (|f'(x_i)| + |f'(x_{i+1})|) \end{aligned}$$

which completes the proof.  $\square$

**Proposition 4.** *Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|^q$  is concave on  $[a, b]$ , for some fixed  $q > 1$ . Then in (4.1), for every division  $d$  of  $[a, b]$ , the trapezoidal error estimate satisfies*

$$|E(f, d)| \leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^2}{4} \left( \left| f' \left( \frac{3x_i + x_{i+1}}{4} \right) \right| + \left| f' \left( \frac{x_i + 3x_{i+1}}{4} \right) \right| \right).$$

*Proof.* The proof is similar to that of Proposition 3 and using Remark 2.  $\square$

**Proposition 5.** *Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|^q$  is concave on  $[a, b]$ , for some fixed  $q \geq 1$ . Then in (4.1), for every division  $d$  of  $[a, b]$ , the trapezoidal error estimate satisfies*

$$|E(f, d)| \leq \frac{1}{8} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left( \left| f' \left( \frac{5x_i + x_{i+1}}{6} \right) \right| + \left| f' \left( \frac{x_i + 5x_{i+1}}{6} \right) \right| \right).$$

*Proof.* The proof is similar to that of Proposition 3 and using Corollary 4. □

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